

Order of Convexity of Integral Transforms and Duality

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Abstract

Recently, Ali et al [2] defined the class $\mathcal{W}_\beta(\alpha, \gamma)$ consisting of functions f which satisfy

$$\Re e^{i\phi} \left((1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) - \beta \right) > 0,$$

for all $z \in E = \{z : |z| < 1\}$ and for $\alpha, \gamma \geq 0$ and $\beta < 1$, $\phi \in \mathbb{R}$ (the set of reals). For $f \in \mathcal{W}_\beta(\alpha, \gamma)$, they discussed the convexity of the integral transform

$$V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,$$

where λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t) dt = 1$. The aim of present paper is to find conditions on $\lambda(t)$ such that $V_\lambda(f)$ is convex of order δ ($0 \leq \delta \leq 1/2$) whenever $f \in \mathcal{W}_\beta(\alpha, \gamma)$. As applications, we study various choices of $\lambda(t)$, related to classical integral transforms.

Key Words: Starlike function, Convex function, Hadamard product, Duality.

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1 Introduction

Let \mathcal{A} denote the class of analytic functions f defined in the open unit disc $E = \{z : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. Let $\mathcal{A}_0 = \{g : g(z) = f(z)/z, f \in \mathcal{A}\}$. Let S be the subclass of \mathcal{A} consisting of univalent functions in E . A function $f \in S$ is said to be starlike or convex, if f maps E conformally onto the domains, respectively, starlike with respect to the origin and convex. The generalization of these two classes are given by the following analytic characterizations :

$$S^*(\beta) = \left\{ f \in \mathcal{A} : \Re \left(\frac{zf'(z)}{f(z)} \right) > \beta, \quad 0 \leq \beta < 1 \right\}$$

$$K(\beta) = \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta, \quad 0 \leq \beta < 1 \right\}.$$

For $\beta = 0$, we usually set $S^*(0) = S^*$ and $K(0) = K$.

For two functions $f(z) = z + a_2z^2 + a_3z^3 + \dots$ and $g(z) = z + b_2z^2 + b_3z^3 + \dots$ in \mathcal{A} , their Hadamard product (or convolution) is the function $f * g$ defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For $f \in \mathcal{A}$, Fournier and Ruscheweyh [8] introduced the operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \quad (1.1)$$

where λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t) dt = 1$. This operator contains some of the well-known operators such as Libera, Bernardi and Komatu as its special cases. This operator has been studied by a number of authors for various choices of $\lambda(t)$ (for example see [1], [4], [6], [8]). Fournier and Ruscheweyh [8] applied the duality theory ([10], [11]) to prove the starlikeness of the linear integral transform $V_\lambda(f)$ when f varies in the class

$$\mathcal{P}(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} |\Re e^{i\phi} (f'(z) - \beta)| > 0, z \in E \right\}.$$

In 1995, Ali and Singh [3] discussed the convexity properties of the integral transform (1.1) for functions f in the class $\mathcal{P}(\beta)$. In 2002, Choi et al. [7] investigated convexity properties of the integral transform (1.1) for functions f in the class

$$\mathcal{P}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} |\Re e^{i\phi} \left((1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) - \beta \right)| > 0, z \in E \right\}.$$

It is evident that the class $\mathcal{P}_\gamma(\beta)$ is closely related to the class $\mathcal{R}_\gamma(\beta)$ defined by

$$\mathcal{R}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} |\Re e^{i\phi} (f'(z) + \gamma z f''(z) - \beta)| > 0, z \in E \right\}.$$

Clearly, $f \in \mathcal{R}_\gamma(\beta)$ if and only if zf' belongs to $\mathcal{P}_\gamma(\beta)$.

In a very recent paper, R.M.ali et al [2] discussed the convexity of the integral transform (1.1) for the functions f in a more general class $\mathcal{W}_\beta(\alpha, \gamma)$

$$\left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} |\Re e^{i\phi} \left((1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) - \beta \right)| > 0, z \in E \right\}. \quad (1.2)$$

Note that $\mathcal{W}_\beta(1, 0) \equiv \mathcal{P}(\beta)$, $\mathcal{W}_\beta(\alpha, 0) \equiv \mathcal{P}_\alpha(\beta)$ and $\mathcal{W}_\beta(1 + 2\gamma, \gamma) \equiv \mathcal{R}_\gamma(\beta)$.

In the present paper, we shall mainly tackle the problem of finding a sharp estimate of the parameter β that ensures $V_\lambda(f)$ to be convex of order δ for $f \in \mathcal{W}_\beta(\alpha, \gamma)$. To prove our result, we shall need the duality theory for convolutions, so we include here some basic concepts and results from this theory. For a subset $\mathcal{B} \subset \mathcal{A}_0$, we define

$$\mathcal{B}^* = \{g \in \mathcal{A}_0 : (f * g)(z) \neq 0, z \in E, \text{ for all } f \in \mathcal{B}\}$$

The set \mathcal{B}^* is called the dual of \mathcal{B} . Further, the second dual of \mathcal{B} is defined as $\mathcal{B}^{**} = (\mathcal{B}^*)^*$. We state below a fundamental result.

Theorem 1.1. Let

$$\mathcal{B} = \left\{ \beta + (1 - \beta) \left(\frac{1 + xz}{1 + yz} \right) : |x| = |y| = 1 \right\}, \quad \beta \in \mathbb{R}, \beta \neq 1.$$

Then, we have

$$(1) \mathcal{B}^{**} = \{g \in \mathcal{A}_0 : \exists \phi \in \mathbb{R} \text{ such that } \Re\{e^{i\phi}(g(z) - \beta)\} > 0, z \in E\}.$$

(2) If Γ_1 and Γ_2 are two continuous linear functionals on \mathcal{B} with $0 \notin \Gamma_2$, then for every $g \in \mathcal{B}^{**}$ we can find $v \in \mathcal{B}$ such that

$$\frac{\Gamma_1(g)}{\Gamma_2(g)} = \frac{\Gamma_1(v)}{\Gamma_2(v)}.$$

The basic reference to this theory is the book by Ruscheweyh [10] (see also [11]).

2 Preliminaries

We follow the notations used in [1]. Let $\mu \geq 0$ and $\nu \geq 0$ satisfy

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma. \quad (2.1)$$

When $\gamma = 0$, then μ is chosen to be 0, in which case, $\nu = \alpha \geq 0$. When $\alpha = 1 + 2\gamma$, (2.1) yields $\mu + \nu = 1 + \gamma = 1 + \mu\nu$, or $(\mu - 1)(1 - \nu) = 0$.

- (i) For $\gamma > 0$, then choosing $\mu = 1$ gives $\nu = \gamma$.
- (ii) For $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha = 1$.

Whenever the particular case $\alpha = 1 + 2\gamma$ will be considered, the values of μ and ν for $\gamma > 0$ will be taken as $\mu = 1$ and $\nu = \gamma$ respectively, while $\mu = 0$ and $\nu = 1 = \alpha$ in the case when $\gamma = 0$.

Next we introduce two auxiliary functions. Let

$$\phi_{\mu,\nu}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n\nu + 1)(n\mu + 1)}{n + 1} z^n, \quad (2.2)$$

and

$$\begin{aligned} \psi_{\mu,\nu}(z) &= \phi_{\mu,\nu}^{-1}(z) = 1 + \sum_{n=1}^{\infty} \frac{n + 1}{(n\nu + 1)(n\mu + 1)} z^n \\ &= \int_0^1 \int_0^1 \frac{ds dt}{(1 - t^\nu s^\mu z)^2}. \end{aligned} \quad (2.3)$$

Here $\phi_{\mu,\nu}^{-1}$ denotes the convolution inverse of $\phi_{\mu,\nu}$ such that $\phi_{\mu,\nu} * \phi_{\mu,\nu}^{-1} = z/(1 - z)$. If $\gamma = 0$, then $\mu = 0$, $\nu = \alpha$, and it is clear that

$$\psi_{0,\alpha}(z) = 1 + \sum_{n=1}^{\infty} \frac{n + 1}{n\alpha + 1} z^n = \int_0^1 \frac{dt}{(1 - t^\alpha z)^2}.$$

If $\gamma > 0$, then $\nu > 0$, $\mu > 0$, and making the change of variables $u = t^\nu$, $v = s^\mu$ results in

$$\psi_{\mu,\nu}(z) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1-uvz)^2} du dv.$$

Thus the function $\psi_{\mu,\nu}$ can be written as

$$\psi_{\mu,\nu}(z) = \begin{cases} \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1-uvz)^2} du dv, & \gamma > 0; \\ \int_0^1 \frac{dt}{(1-t^\alpha z)^2}, & \gamma = 0, \alpha > 0. \end{cases} \quad (2.4)$$

Let q be the solution of the initial value problem

$$\frac{d}{dt} \left(t^{1/\nu} q(t) \right) = \begin{cases} \frac{1}{\mu\nu} t^{1/\nu-1} \int_0^1 \frac{(1-\delta)-(1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\mu-1} ds, & \gamma > 0, \\ \frac{1}{\alpha} \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} t^{1/\alpha-1}, & \gamma = 0, \alpha > 0, \end{cases} \quad (2.5)$$

satisfying $q(0) = 1$.

Solving the differential equation (2.5), we have

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta)-(1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} ds dw. \quad (2.6)$$

In particular,

$$q_\alpha(t) = \frac{1}{\alpha} \int_0^1 \frac{(1-\delta)-(1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\alpha-1} ds, \quad \gamma = 0, \alpha > 0. \quad (2.7)$$

Further let

$$\Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}} dx, \quad \nu > 0, \quad (2.8)$$

and

$$\Pi_{\mu,\nu}(t) = \begin{cases} \int_t^1 \Lambda_\nu(x) x^{1/\nu-1-1/\mu} dx, & \gamma > 0, \\ \Lambda_\alpha(t), & \gamma = 0, (\mu = 0, \nu = \alpha > 0). \end{cases} \quad (2.9)$$

For the function $\Pi_{\mu,\nu}(t)$, we define

$$\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta) = \begin{cases} \Re \int_0^1 t^{1/\mu-1} \Pi_{\mu,\nu}(t) \left[h'_\delta(tz) - \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt, & \gamma > 0, \\ \Re \int_0^1 t^{1/\alpha-1} \Pi_{0,\alpha}(t) \left[h'_\delta(tz) - \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt, & \gamma = 0, \end{cases} \quad (2.10)$$

where $h_\delta(z)$ is defined as

$$h_\delta(z) = \frac{z \left(1 + \frac{\epsilon+2\delta-1}{2-2\delta} z \right)}{(1-z)^2}, \quad |\epsilon| = 1. \quad (2.11)$$

With these notations, we are now in a position to state our first result, which generalizes many earlier results in this direction.

3 Main results

Theorem 3.1. Let $\mu \geq 0$, $\nu \geq 0$ satisfy (2.1). Define $\beta < 1$ by

$$\frac{\beta - \frac{1}{2}}{(1 - \beta)} = - \int_0^1 \lambda(t) q(t) dt, \quad (3.1)$$

where $q(t)$ is the solution of the initial-value problem (2.5). Further for $\Lambda_\nu(t)$ and $\Pi_{\mu,\nu}(t)$ defined by (2.8) and (2.9) respectively, assume that $t^{1/\nu}\Lambda_\nu(t) \rightarrow 0$, and $t^{1/\nu}\Pi_{\mu,\nu}(t) \rightarrow 0$ as $t \rightarrow 0^+$. Then for $\delta \in [0, \frac{1}{2}]$, $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \subset K(\delta)$ if and only if $\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta) \geq 0$, where $\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta)$ and h_δ are defined by equations (2.10) and (2.11) respectively.

Proof. As the case $\gamma = 0$ ($\mu = 0$, $\nu = \alpha$) corresponds to the Theorem 2.3 in [5], so we will prove the result only when $\gamma > 0$.

Let

$$H(z) = (1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z).$$

Since $\mu + \nu = \alpha - \gamma$ and $\mu\nu = \gamma$, therefore

$$\begin{aligned} H(z) &= (1 + \gamma - (\alpha - \gamma)) \frac{f(z)}{z} + (\alpha - \gamma - \gamma) f'(z) + \gamma z f''(z) \\ &= (1 + \mu\nu - \mu - \nu) \frac{f(z)}{z} + (\mu + \nu - \mu\nu) f'(z) + \mu\nu z f''(z). \end{aligned}$$

Writing $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we obtain from (2.2)

$$H(z) = 1 + \sum_{n=1}^{\infty} a_{n+1} (n\nu + 1)(n\mu + 1) z^n = f'(z) * \phi_{\mu,\nu}(z), \quad (3.2)$$

and (2.3) gives that

$$f'(z) = H(z) * \psi_{\mu,\nu}(z). \quad (3.3)$$

Now, for $f \in \mathcal{W}_\beta(\alpha, \gamma)$, we have

$$\Re \left\{ e^{i\phi} \frac{H(z) - \beta}{1 - \beta} \right\} > 0.$$

Thus, in the view of the Theorem 1.1, we may confine ourselves to functions $f \in \mathcal{W}_\beta(\alpha, \gamma)$ for which

$$H(z) = \beta + (1 - \beta) \left(\frac{1 + xz}{1 + yz} \right), \quad |x| = |y| = 1.$$

Thus (3.3) gives

$$f'(z) = \left((1 - \beta) \frac{1 + xz}{1 + yz} + \beta \right) * \psi_{\mu,\nu}(z), \quad (3.4)$$

and therefore

$$\frac{f(z)}{z} = \frac{1}{z} \int_0^z \left((1 - \beta) \frac{1 + xw}{1 + yw} + \beta \right) dw * \psi(z). \quad (3.5)$$

Here $\psi := \psi_{\mu,\nu}$.

A well-known result from the theory of convolutions [9, Pg 94] (also see [11]) states that

$$F \in K(\delta) \Leftrightarrow \frac{1}{z}(zF' * h_\delta)(z) \neq 0, \quad z \in E,$$

where

$$h_\delta(z) = \frac{z \left(1 + \frac{\epsilon+2\delta-1}{2-2\delta}z\right)}{(1-z)^2}, \quad |\epsilon| = 1.$$

Hence $F \in K(\delta)$ if and only if

$$0 \neq \frac{1}{z}(V_\lambda(f)(z) * zh'_\delta(z)) = \frac{1}{z} \left[\int_0^1 \lambda(t) \frac{f(tz)}{t} dt * zh'_\delta(z) \right] = \int_0^1 \frac{\lambda(t)}{1-tz} dt * \frac{f(z)}{z} * h'_\delta(z)$$

Using (3.5), we have

$$\begin{aligned} 0 &\neq \int_0^1 \frac{\lambda(t)}{1-tz} dt * \left[\frac{1}{z} \int_0^z \left((1-\beta) \frac{1+xw}{1+yw} + \beta \right) dw * \psi(z) \right] * h'_\delta(z) \\ &= \int_0^1 \frac{\lambda(t)}{1-tz} dt * h'_\delta(z) * \left[\frac{1}{z} \int_0^z \left((1-\beta) \frac{1+xw}{1+yw} + \beta \right) dw \right] * \psi(z) \\ &= \int_0^1 \lambda(t) h'_\delta(tz) dt * (1-\beta) \left[\frac{1}{z} \int_0^z \left(\frac{1+xw}{1+yw} + \frac{\beta}{(1-\beta)} \right) dw \right] * \psi(z) \\ &= (1-\beta) \left[\int_0^1 \lambda(t) h'_\delta(tz) dt + \frac{\beta}{(1-\beta)} \right] * \frac{1}{z} \int_0^z \frac{1+xw}{1+yw} dw * \psi(z) \\ &= (1-\beta) \left[\int_0^1 \lambda(t) \left(\frac{1}{z} \int_0^z h'_\delta(tw) dw \right) dt + \frac{\beta}{(1-\beta)} \right] * \frac{1+xz}{1+yz} * \psi(z). \end{aligned}$$

This holds if and only if [11, p. 23]

$$\begin{aligned} &\Re(1-\beta) \left[\int_0^1 \lambda(t) \left(\frac{1}{z} \int_0^z h'_\delta(tw) dw \right) dt + \frac{\beta}{(1-\beta)} \right] * \psi(z) \geq \frac{1}{2}, \\ \Leftrightarrow &\Re(1-\beta) \left[\int_0^1 \lambda(t) \left(\frac{1}{z} \int_0^z h'_\delta(tw) dw \right) dt + \frac{\beta}{(1-\beta)} - \frac{1}{2(1-\beta)} \right] * \psi(z) \geq 0, \\ \Leftrightarrow &\Re \left[\int_0^1 \lambda(t) \left(\frac{1}{z} \int_0^z h'_\delta(tw) dw \right) dt + \frac{\beta - \frac{1}{2}}{(1-\beta)} \right] * \psi(z) \geq 0, \\ \Leftrightarrow &\Re \left[\int_0^1 \lambda(t) \left(\frac{1}{z} \int_0^z h'_\delta(tw) dw - q(t) \right) dt \right] * \psi(z) \geq 0, \quad (\text{using (3.1)}) \end{aligned}$$

$$\begin{aligned} \Leftrightarrow &\Re \left[\int_0^1 \lambda(t) (h'_\delta(tz) - q(t)) dt \right] * \frac{1}{z} \int_0^z \psi(w) dw \geq 0, \\ \Leftrightarrow &\Re \left[\int_0^1 \lambda(t) (h'_\delta(tz) - q(t)) dt \right] * \sum_{n=0}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} \geq 0, \quad (\text{using (2.3)}) \\ \Leftrightarrow &\Re \int_0^1 \lambda(t) \left(\sum_{n=0}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} * h'_\delta(tz) - q(t) \right) dt \geq 0, \\ \Leftrightarrow &\Re \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{d\eta d\zeta}{(1-\eta^\nu \zeta^\mu z)} * h'_\delta(tz) - q(t) \right) dt \geq 0, \\ \Leftrightarrow &\Re \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 h'_\delta(tz \eta^\nu \zeta^\mu) d\eta d\zeta - q(t) \right) dt \geq 0, \end{aligned}$$

which can also be written as

$$\Re \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{1}{\mu\nu} h'_\delta(tzuv) u^{1/\nu-1} v^{1/\mu-1} dv du - q(t) \right) dt \geq 0.$$

Writing $w = tu$, we get

$$\Re \int_0^1 \frac{\lambda(t)}{t^{1/\nu}} \left[\int_0^t \int_0^1 h'_\delta(wzv) w^{1/\nu-1} v^{1/\mu-1} dv dw - \mu\nu t^{1/\nu} q(t) \right] dt \geq 0.$$

An integration by parts with respect to t and (2.5) gives

$$\Re \int_0^1 \Lambda_\nu(t) \left[\int_0^1 h'_\delta(tzv) t^{1/\nu-1} v^{1/\mu-1} dv - t^{1/\nu-1} \int_0^1 \frac{1-\delta-(1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\mu-1} ds \right] dt \geq 0.$$

Again writing $w = vt$ and $\eta = st$ above inequality reduces to

$$\Re \int_0^1 \Lambda_\nu(t) t^{1/\nu-1/\mu-1} \left[\int_0^t h'_\delta(wz) w^{1/\mu-1} dw - \int_0^t \frac{1-\delta-(1+\delta)\eta}{(1-\delta)(1+\eta)^3} \eta^{1/\mu-1} d\eta \right] dt \geq 0,$$

which after integration by parts with respect to t yields

$$\Re \int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu-1} \left[h'_\delta(tz) - \frac{1-\delta-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt \geq 0.$$

Thus $F \in K(\delta)$ if and only if $\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta) \geq 0$.

Finally, to prove the sharpness, let $f \in \mathcal{W}_\beta(\alpha, \gamma)$ be of the form for which

$$(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) = \beta + (1-\beta)\frac{1+z}{1-z}.$$

Using a series expansion we obtain that

$$f(z) = z + 2(1-\beta) \sum_{n=1}^{\infty} \frac{1}{(n\nu+1)(n\mu+1)} z^{n+1}.$$

Thus

$$F(z) = V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt = z + 2(1-\beta) \sum_{n=1}^{\infty} \frac{\tau_n}{(n\nu+1)(n\mu+1)} z^{n+1},$$

where $\tau_n = \int_0^1 \lambda(t) t^n dt$. From (2.5), it is a simple exercise to write $q(t)$ in a series expansion as

$$q(t) = 1 + \frac{1}{1-\delta} \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(n+1-\delta)}{(n\nu+1)(n\mu+1)} t^n. \quad (3.6)$$

Now, by (3.1) and (3.6), we have

$$\begin{aligned} \frac{\beta - \frac{1}{2}}{1-\beta} &= - \int_0^1 \lambda(t) q(t) dt \\ &= - \int_0^1 \lambda(t) \left[1 + \frac{1}{1-\delta} \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(n+1-\delta)}{(n\nu+1)(n\mu+1)} t^n \right] dt \\ &= -1 - \frac{1}{1-\delta} \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(n+1-\delta)}{(n\nu+1)(n\mu+1)} \int_0^1 \lambda(t) t^n dt. \end{aligned}$$

Therefore

$$\frac{1}{2(1-\beta)} = -\frac{1}{1-\delta} \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(n+1-\delta)\tau_n}{(n\nu+1)(n\mu+1)}. \quad (3.7)$$

Finally, we see that

$$F'(z) = 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(n+1)\tau_n}{(n\nu+1)(n\mu+1)} z^n.$$

Therefore

$$(zF'(z))' = 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(n+1)^2\tau_n}{(n\nu+1)(n\mu+1)} z^n.$$

For $z = -1$, we have

$$\begin{aligned} (zF')'(-1) &= 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^2\tau_n}{(n\nu+1)(n\mu+1)} \\ &= 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(n+1-\delta)\tau_n}{(n\nu+1)(n\mu+1)} + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n \delta (n+1)\tau_n}{(n\nu+1)(n\mu+1)} \\ &= 1 - (1-\delta) + \delta 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)\tau_n}{(n\nu+1)(n\mu+1)} \quad (\text{Using (3.7)}) \\ &= \delta \left(1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)\tau_n}{(n\nu+1)(n\mu+1)} \right) \\ &= \delta F'(-1). \end{aligned}$$

Thus $(zF'(z))'/F'(z)$ at $z = -1$ equals δ . This implies that the result is sharp for the order of convexity.

4 Consequences of Theorem 3.1

To obtain a sufficient condition for the convexity of order δ of the integral transform (1.1) by a much easier method, we present the following theorem.

Theorem 4.1. Let $\Lambda_\nu(t)$, $\Pi_{\mu,\nu}(t)$ be integrable on $[0,1]$ and positive on $(0,1)$. Also, suppose that $t^{1/\nu}\Lambda_\nu(t) \rightarrow 0$, and $t^{1/\nu}\Pi_{\mu,\nu}(t) \rightarrow 0$ as $t \rightarrow 0^+$. Assume further that $\mu \geq 1$ and

$$\frac{\left(-t\Pi'_{\mu,\nu}(t) + \left(1 - \frac{1}{\mu}\right)\Pi_{\mu,\nu}(t)\right)}{(1+t)(1-t)^{1+2\delta}} \text{ is decreasing on } (0,1). \quad (4.1)$$

For $\delta \in [0, 1/2]$, if β satisfies (3.1), then $V_\lambda(f) \in K(\delta)$ for $f \in \mathcal{W}_\beta(\alpha, \gamma)$.

Proof. For $\gamma > 0$, integration by parts with respect to t yields

$$\begin{aligned} & \int_0^1 t^{\frac{1}{\mu}-1} \Pi_{\mu,\nu}(t) \left(\Re(h'_\delta(tz)) - \frac{1-\delta-(1+\delta)t}{(1-\delta)(1+t)^3} \right) dt \\ &= \int_0^1 t^{\frac{1}{\mu}-1} \Pi_{\mu,\nu}(t) \frac{d}{dt} \left(\Re \frac{h_\delta(tz)}{z} - \frac{t(1-\delta(1+t))}{(1-\delta)(1+t)^2} \right) dt \\ &= \int_0^1 t^{\frac{1}{\mu}-1} \left(-t\Pi'_{\mu,\nu}(t) + \left(1 - \frac{1}{\mu}\right)\Pi_{\mu,\nu}(t) \right) \left(\Re \frac{h_\delta(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2} \right) dt. \end{aligned}$$

Also for $\mu \geq 1$, the function $t^{1/\mu-1}$ is decreasing on $(0,1)$. Thus, the condition (4.1) along with Theorem 1 from [8] yields

$$\int_0^1 t^{\frac{1}{\mu}-1} \Pi_{\mu,\nu}(t) \left(\Re(h'_\delta(tz)) - \frac{1-\delta-(1+\delta)t}{(1-\delta)(1+t)^3} \right) dt > 0.$$

Thus, an application of Theorem 3.1 evidently leads to the desired result. \square

Below, we obtain the conditions to ensure convexity of $V_\lambda(f)$. As defined in (2.8) and (2.9), for $\gamma > 0$,

$$\Pi_{\mu,\nu}(t) = \int_t^1 \Lambda_\nu(x) x^{1/\nu-1-1/\mu} dx, \text{ and } \Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}} dx.$$

In order to apply Theorem 4.1, we have to prove that the function

$$k(t) = \frac{\left(t^{\frac{1}{\nu}-\frac{1}{\mu}} \Lambda_\nu(t) + \left(1 - \frac{1}{\mu}\right) \Pi_{\mu,\nu}(t)\right)}{(1+t)(1-t)^{1+2\delta}} := \frac{p(t)}{(1+t)(1-t)^{1+2\delta}}$$

is decreasing in $(0,1)$. Since $k(t) > 0$ and

$$\begin{aligned} \frac{k'(t)}{k(t)} &= \frac{p'(t)}{p(t)} + \frac{2(t+\delta(1+t))}{1-t^2} \\ &= \frac{2(t+\delta(1+t))}{(1-t^2)p(t)} \left[\frac{(1-t^2)p'(t)}{2(t+\delta(1+t))} + p(t) \right] = \frac{2(t+\delta(1+t))}{(1-t^2)p(t)} [q(t)] \text{ (say)}. \end{aligned}$$

Thus to prove that $k'(t) \leq 0$, it is enough to prove that $q(t) \leq 0$. Since $q(1) = 0$, so it remains to show that $q(t)$ is increasing over $(0,1)$. Now

$$q'(t) = \frac{(1+t)}{2(t+\delta(1+t))^2} [(1-t)(t+\delta(1+t))p''(t) - (1-t-\delta(1+t))(1+2\delta)p'(t)].$$

So, $q'(t) \geq 0$ for $t \in (0,1)$ is equivalent to the inequality $r(t) \geq 0$, where

$$r(t) = (1-t)(t+\delta(1+t))p''(t) - (1-t-\delta(1+t))(1+2\delta)p'(t)$$

By using the idea similar to the one used to prove Theorem 3.1 in [6], we can write

$$r(t) = -\lambda(t)t^{1-\frac{1}{\mu}} \left[\left(\frac{1}{\nu} - \frac{1}{\mu} - 1 \right) X(t) + Z(t) + \frac{t\lambda'(t)}{\lambda(t)} X(t) \right] + \left[\left(\frac{1}{\nu} - \frac{1}{\mu} - 1 \right) X(t) + Z(t) \right] \left(\frac{1}{\nu} - 1 \right) t^{\frac{1}{\nu}-\frac{1}{\mu}-1} \int_t^1 A(s) ds \quad (4.2)$$

where,

$$\begin{aligned} A(t) &= \lambda(t)t^{-1/\nu}, \\ X(t) &= (1-t)(t+\delta(1+t)), \\ Z(t) &= -t(1-t-\delta(1+t))(1+2\delta). \end{aligned} \quad (4.3)$$

Clearly, $A(t) > 0$ and $X(t) > 0$ for all $t \in (0,1)$.

Thus, $r(t)$ is non-negative if

$$\left(\frac{1}{\nu} - \frac{1}{\mu} - 1 \right) X(t) + Z(t) + \frac{t\lambda'(t)}{\lambda(t)} X(t) \leq 0 \text{ and } \left[\left(\frac{1}{\nu} - \frac{1}{\mu} - 1 \right) X(t) + Z(t) \right] \left(\frac{1}{\nu} - 1 \right) \geq 0. \quad (4.4)$$

Since $\nu \geq 1$, we can rewrite the condition (4.4) as follows :

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} - \left(\frac{X(t) + Z(t)}{X(t)} \right) \quad \text{and} \quad \frac{1}{\nu} - \frac{1}{\mu} - 2 \leq - \left(\frac{X(t) + Z(t)}{X(t)} \right). \quad (4.5)$$

In view of the fact that $X(t) + Z(t)$ and $X(t)$ are non-negative on $(0,1)$, the above inequality further reduces to

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} \quad \text{and} \quad \frac{1}{\nu} - \frac{1}{\mu} - 2 \leq 0. \quad (4.6)$$

For $\mu \geq 1$, condition (2.1) implies $\nu \geq \mu \geq 1$. Thus, condition (4.6) implies that $r(t)$ is non-negative if

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \quad (4.7)$$

These conditions leads to the following theorem.

Theorem 4.2. Assume that both $\Lambda_\nu(t)$, $\Pi_{\mu,\nu}(t)$ are integrable on $[0,1]$ and positive on $(0,1)$. Let $\lambda(t)$ be a non-negative real-valued integrable function on $[0,1]$ and satisfy the condition

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \quad (4.8)$$

Let $f \in \mathcal{W}_\beta(\alpha, \gamma)$ and $\beta < 1$ with

$$\frac{\beta - \frac{1}{2}}{(1 - \beta)} = - \int_0^1 \lambda(t)q(t)dt,$$

where $q(t)$ is defined by (2.6). Then $F(z) = V_\lambda(f)(z) \in K(\delta)$ for $\delta \in [0, 1/2]$. The conclusion does not hold for smaller values of β .

On the other hand, when $\gamma = 0$ ($\mu = 0$, $\nu = \alpha > 0$), so we get the following result.

Theorem 4.3. Let $\lambda(t)$ be a non-negative real-valued integrable function on $[0,1]$. Assume that both $\Lambda_\alpha(t)$, $\Pi_{0,\alpha}(t)$ are integrable on $[0,1]$ and positive on $(0,1)$. Let $\lambda(1) = 0$ and λ satisfies the condition

$$t\lambda''(t) - \frac{1}{\alpha}\lambda'(t) \geq 0, \quad \alpha \geq 1. \quad (4.9)$$

Let $f \in \mathcal{W}_\beta(\alpha, \gamma)$ and $\beta < 1$ with

$$\frac{\beta - \frac{1}{2}}{(1 - \beta)} = - \int_0^1 \lambda(t)q_\alpha(t)dt,$$

where $q_\alpha(t)$ is defined by (2.7) with $\delta \in [0, 1/2]$. Then $F(z) = V_\lambda(f)(z) \in K(\delta)$. The conclusion does not hold for smaller values of β .

Proof. As in Theorem 3.1, for $\gamma = 0$ and $f \in \mathcal{W}_\beta(\alpha, \gamma)$, we have $V_\lambda(f)(z) \in K(\delta)$ if

$$\int_0^1 t^{\frac{1}{\alpha}-1} \Pi_{0,\alpha}(t) \left(\Re(h'_\delta(tz)) - \frac{1 - \delta - (1 + \delta)t}{(1 - \delta)(1 + t)^3} \right) dt > 0,$$

which is equivalent to

$$\int_0^1 t^{\frac{1}{\alpha}-1} \left(t^{1-\frac{1}{\alpha}} \lambda(t) + \left(1 - \frac{1}{\alpha} \right) \Lambda_\alpha(t) \right) \left(\Re \frac{h_\delta(tz)}{tz} - \frac{1 - \delta(1 + t)}{(1 - \delta)(1 + t)^2} \right) dt > 0.$$

Since $t^{\frac{1}{\alpha}-1}$ is decreasing on $(0,1)$ for $\alpha \geq 1$, thus to apply Theorem 1 in [8], it is enough to show that

$$p(t) = \frac{t^{1-\frac{1}{\alpha}}\lambda(t) + (1 - \frac{1}{\alpha})\Lambda_\alpha(t)}{(1+t)(1-t)^{1+2\delta}} := \frac{k(t)}{(1+t)(1-t)^{1+2\delta}}$$

is decreasing on $(0,1)$. Here, logarithmic differentiation implies that

$$\frac{p'(t)}{p(t)} = \frac{2(t + \delta(1+t))}{(1-t^2)k(t)} \left[\frac{(1-t^2)k'(t)}{2(t + \delta(1+t))} + k(t) \right].$$

Since $p(t) > 0$ for $\alpha \geq 1$, thus to prove that $p'(t) \leq 0$ on $(0,1)$ it remains to show that

$$r(t) = k(t) + \frac{(1-t^2)k'(t)}{2(t + \delta(1+t))} \leq 0.$$

Since $r(1) = 0$, so $r(t) \leq 0$ if $r(t)$ is increasing on $(0,1)$. Thus, $r'(t)$ is non-negative if

$$\frac{t^{\frac{-1}{\alpha}}(1+t)}{2(t + \delta(1+t))} \left\{ X(t)t\lambda''(t) + \left[\left(1 - \frac{1}{\alpha}\right)X(t) + Z(t) \right] \lambda'(t) \right\} \geq 0,$$

where $X(t)$ and $Z(t)$ are as defined in (4.3). Further simplification yields that

$$t\lambda''(t) + \left(\frac{X(t) + Z(t)}{X(t)} - \frac{1}{\alpha} \right) \lambda'(t) \geq 0.$$

Since, $X(t)$ and $X(t) + Z(t)$ are non-negative in $(0,1)$, thus $r'(t) \geq 0$ is equivalent to

$$t\lambda''(t) - \frac{1}{\alpha}\lambda'(t) \geq 0, \quad \alpha \geq 1,$$

which completes the proof.

Remarks 4.4. Observe that results in [2] can be obtained from our results by setting $\delta = 0$.

5 Applications

In this section, we apply Theorem 4.2 and Theorem 4.3 to obtain certain results regarding convexity of well-known integral operators. The proofs of the following results run on the same lines as given in [2] and hence omitted.

Consider λ to be defined as

$$\lambda(t) = (1+c)t^c, \quad c > -1.$$

Then the integral transform

$$F_c(z) = V_\lambda(f)(z) = (1+c) \int_0^1 t^{c-1} f(tz) dt, \quad c > -1, \quad (5.1)$$

is the well-known Bernardi integral operator. The classical Alexander and Libera transforms are special cases of (5.1) with $c = 0$ and $c = 1$ respectively. For this special case of λ , the following result holds.

Theorem 5.1. Let $c > -1$ and $0 < \gamma \leq \alpha \leq 1 + 2\gamma$. Let $\beta < 1$ satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = -(1 + c) \int_0^1 t^c q(t) dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1 - \delta) - (1 + \delta)swt}{(1 - \delta)(1 + swt)^3} s^{1/\mu-1} w^{1/\nu-1} ds dw.$$

Then for $\delta \in [0, 1/2]$, we have $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \subset K(\delta)$ provided c satisfies the condition :

$$c \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \quad (5.2)$$

The value of β is sharp.

Writing $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 5.1 gives the following criteria of convexity :

Corollary 5.2. Let $-1 < c \leq 3 - 1/\gamma$ and $\gamma \geq 1$. Let $\beta < 1$ satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = -(1 + c) \int_0^1 t^c q(t) dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1 - \delta) - (1 + \delta)swt}{(1 - \delta)(1 + swt)^3} s^{1/\mu-1} w^{1/\nu-1} ds dw.$$

Then for $\delta \in [0, 1/2]$, we have $V_\lambda(\mathcal{R}_\beta(\gamma)) \subset K(\delta)$. The value of β is sharp.

Further, letting $\gamma = 1$ and $c = 0$ in Corollary 5.2, we have

Corollary 5.3. Let $\beta < 1$ satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = \frac{1}{1 - \delta} \left(\delta \frac{\pi^2}{12} - \log 2 \right)$$

If $f \in \mathcal{R}_\beta(1)$, then Alexander transform $F_0(z) \equiv A[f](z) = \int_0^1 \frac{f(tz)}{t} dt$ is convex of order δ where $\delta \in [0, 1/2]$. The value of β is sharp.

Remark 5.4. 1. For $\delta = 0$,

$$\beta_0 = \frac{1 - 2 \log 2}{2 - 2 \log 2} = -0.629 \dots$$

Then, for f satisfying

$$\Re e^{i\phi} (f'(z) + zf''(z) - \beta) > 0, \quad z \in E,$$

Alexander transform $A[f]$ is convex. It has been shown in [8] that β_0 is the best possible bound here.

2. We note that for $\delta = 1/2$, $\beta_{1/2} = 0.590 \dots$. Then, for f satisfying

$$\Re e^{i\phi} (f'(z) + zf''(z) - \beta) > 0, \quad z \in E,$$

Alexander transform $A[f]$ is convex of order $\frac{1}{2}$.

While, the case $c = 0$ in Theorem 5.1 yields yet another interesting result, which we state as a theorem.

Theorem 5.5. Let $0 < \gamma \leq \alpha \leq 1 + 2\gamma$. If $F \in \mathcal{A}$ satisfies

$$\Re(F'(z) + \alpha z F''(z) + \gamma z^2 F'''(z)) > \beta, \quad z \in E,$$

and $\beta < 1$ satisfies

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 q(t) dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} ds dw,$$

then for $\delta \in [0, 1/2]$, F belongs to $K(\delta)$. The value of β is sharp.

To state our next theorem, we define

$$\lambda(t) = \begin{cases} (a+1)(b+1) \frac{t^a(1-t^{b-a})}{b-a}, & b \neq a; \\ (a+1)^2 t^a \log(1/t), & b = a, \end{cases} \quad (5.3)$$

where $b > -1$ and $a > -1$.

Then,

$$V_\lambda(f)(z) = G_f(a, b; z) = \begin{cases} \frac{(a+1)(b+1)}{b-a} \int_0^1 t^{a-1} (1-t^{b-a}) f(tz) dt, & b \neq a; \\ (a+1)^2 \int_0^1 t^{a-1} \log(1/t) f(tz) dt, & b = a. \end{cases}$$

Theorem 5.6. Let $b > -1$, $a > -1$ and $0 < \gamma \leq \alpha \leq 1 + 2\gamma$. Let $\beta < 1$ satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \lambda(t) q(t) dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} ds dw.$$

and $\lambda(t)$ is defined by (5.3). If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the convolution operator $G_f(a, b; z)$ belongs to $K(\delta)$ with $\delta \in [0, 1/2]$ if

$$a \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \quad (5.4)$$

The value of β is sharp.

Substituting $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 5.1, gives the following result :

Corollary 5.7. Let $b > -1$, $-1 < a \leq 3 - 1/\gamma$ and $\gamma \geq 1$. Let $\beta < 1$ satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \lambda(t) q(t) dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} dsdw.$$

and $\lambda(t)$ is defined by (5.3). If $f \in \mathcal{R}_\beta(\gamma)$, then the convolution operator $G_f(a, b; z)$ belongs to $K(\delta)$ with $\delta \in [0, 1/2]$. The value of β is sharp.

While for $\gamma = 0$, with an application of Theorem 4.3, we get the following result :

Theorem 5.8. Let $b > -1$, $a > -1$ and $\alpha \geq 1$. Let $\beta < 1$ satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \lambda(t) q_\alpha(t) dt,$$

where q_α is given by

$$q_\alpha(t) = \frac{1}{\alpha} \int_0^1 \frac{(1-\delta) - (1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\alpha-1} ds$$

and $\lambda(t)$ is defined by (5.3). If $f \in \mathcal{P}_\beta(\alpha)$, then the convolution operator $G_f(a, b; z)$ belongs to $K(\delta)$ with $\delta \in [0, 1/2]$ if one of the following conditions holds :

- (i) $-1 < a \leq 0$ and $a = b$, or
- (ii) $-1 < a \leq 0$ and $-1 < a < b \leq 1 + 1/\alpha$.

The value of β is sharp.

Now, we define

$$\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a (\log(1/t))^{p-1}, \quad a > -1, \quad p \geq 0.$$

In this case, V_λ reduces to the Komatu operator [9]

$$V_\lambda(f)(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log \left(\frac{1}{t} \right) \right)^{p-1} t^{a-1} f(tz) dt, \quad a > -1, \quad p \geq 0.$$

For $p = 1$ Komatu operator gives the Bernardi integral operator. For this λ , the following result holds.

Theorem 5.9. Let $a > p - 2 > -1$ and $0 < \gamma \leq \alpha \leq 1 + 2\gamma$. Let $\beta < 1$ satisfy

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \lambda(t) q(t) dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} dsdw.$$

If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function

$$\Phi_p(a; z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log \left(\frac{1}{t} \right) \right)^{p-1} t^{a-1} f(tz) dt$$

belongs to $K(\delta)$ with $\delta \in [0, 1/2]$ if

$$a \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \quad \nu \geq \mu \geq 1. \quad (5.5)$$

The value of β is sharp.

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